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THE SHAPE OF THE SURFACE OF A FLUID UNDER CONDITIONS OF WEIGHTLESSNESS†

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The problem of the shape of the surface of a weakly deformed, axially symmetric drop which is in contact with solid parallel walls is considered. A solution of the problem by asymptotic methods is given in the cases of complete wettability and partial wettability.

The shape of the surface of a fluid, under conditions of weightlessness and under the action of surface tension forces, when in contact with solid walls, is determined in the general case by analytic-numerical methods [1].

Investigations have been carried out into the qualitative nature of the shape of the free surface [2, 3].

1. We consider the problem of the equilibrium shape of a fluid which is confined between solid parallel walls under conditions of weightlessness and is acted upon by surface-tension forces in the case of complete non-wettability (mercury between glass walls, for example). This problem is equivalent to determining the shape of a body of specified volume with minimum area of a surface with specified boundaries.

Let a fluid of specified volume V, in the shape of a sphere of radius R, be compressed between parallel planes to a value of the thickness $2a$ $(a < R)$. We shall study this problem assuming that the parameter $(R/a-1)$ is small.

Let us introduce a system of coordinates xyz with its origin at the centre of the drop and with the x-axis directed along the normal to the wall (Fig. 1).

By virtue of the symmetry of the boundary conditions, the free surface of the fluid will be a surface of revolution. Let us write an expression for the area of this surface and the volume of the fluid

$$S = 4\pi\int_0^a F(y,y')dx, \quad V = 2\pi\int_0^a G(y)dx$$

$$F(y,y') = y(1+y'^2)^{1/2}, \quad G(y) = y^2, \quad y' = dy/dx \tag{1.1}$$

The Euler–Lagrange equation for the variational isoperimetric problem under consideration

$$\frac{\partial\Phi}{\partial y} - \frac{d}{dx}\frac{\partial\Phi}{\partial y'} = 0, \quad \Phi = F + \lambda G \tag{1.2}$$

(λ is a Lagrange multiplier) has the form

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$$y(1+y'^2)^{-3/2}y'' - (1+y'^2)^{-1/2} - 2\lambda y = 0 \tag{1.3}$$

A constraint, which forbids them from passing through the surface of the walls, is imposed on the class of permissible curves, among which the extremal is required. According to the theory of one-sided variations, this leads to the condition that they touch the wall [4]. By virtue of the symmetry of the problem, we shall subsequently only be interested in the segment of the extremal which lies in the first quadrant. Since the fact that the derivative goes to infinity when $x = a$ in Cartesian coordinates interferes with the solution of the variational problem, we shall change to polar coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ in Eq. (1.3).

We obtain an equation which has a solution $\rho = a$ when $\lambda = -1/a$ and a first integral

$$\rho' = \frac{\rho(Q\sin\varphi + P\cos\varphi)}{Q\cos\varphi - P\sin\varphi} \tag{1.4}$$

$$\rho' = d\rho/d\varphi, \quad Q = (\rho^2\sin^2\varphi - P^2)^{1/2}, \quad P = \lambda\rho^2\sin^2\varphi + c_1$$

$$c_1 = \text{const}$$

The choice of the sign in front of the expression for P is made from the condition for an extremal to be produced (when $c_1 \neq 0$) from an arc of a circle (when $c_1 = 0$). (Equation (1.4) also has a solution $\rho = a$, if $c_1 = 0$ and $\lambda = -1/a$.) The constant of integration c_1 can be expressed in terms of the parameter λ and the quantity $h = \rho(\pi/2)$, that is, the radius of the equator of the drop (Fig. 1), by using the boundary condition on the left end of the extremal $\rho'(\pi/2) = 0$, $c_1 = -h(1+\lambda h)$.

It can be verified that the discriminant curve

$$\rho = \{[1 - 2\lambda c_1 + (1 - 4\lambda c_1)^{1/2}]/(2\lambda^2)\}^{1/2}/\sin\varphi = h/\sin\varphi \tag{1.5}$$

(that is, the line $y = h$) is a solution of Eq. (1.4) and the band lying between the lines $y = h$ and $y = 0$ is after the extremal. $\Phi_{y'y'} = y(1+y'^2)^{-3/2} \geqslant 0$ for all points from the field of the extremal, that is, the Legendre condition regarding a strong minimum [4] is satisfied. The right-hand boundary condition is that the extremal should touch the wall. This condition is obtained by eliminating ρ' from Eq. (1.4) and the equalities $p' = d(a/\cos\varphi)/d\varphi = a\sin\varphi/\cos^2\varphi$. After the change of variables $\sin^2\varphi = w$ and some simplifications, it takes the form

$$(1+w)^2[c_1(1-w) + \lambda a^2 w]^4 - 2a^2w^2(2-w)(1-w^2)[c_1(1-w) + \lambda a^2 w]^2 + \tag{1.6}$$
$$+ a^4 w^6 (1-w)^2 = 0$$

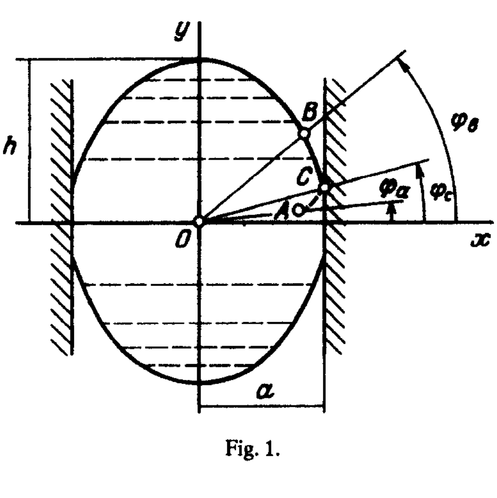

Fig. 1.

Putting $c_1 = \varepsilon a$, $\lambda = -1/a + \varepsilon l_1$, where ε and l_1 are new constants, let us consider the problem of the solvability of Eq. (1.6) for w in the neighbourhood of the value $\varepsilon = 0$. After arranging the terms in order in powers of w, Eq. (1.6) takes the form

$$\begin{aligned}&\varepsilon^4 - 4\varepsilon^3[1+\varepsilon(\tfrac{1}{2} - al_1)]w + \varepsilon^2[2+\varepsilon(4-al_1)+\varepsilon^2(1+6l_2^2-8l_2)]w^2 + \varepsilon[8-4l_3^3+\varepsilon(8l_2+\\&+12l_3^2+2)-4\varepsilon^2 l_3]w^3 + [l_3^4 + 8\varepsilon l_3(1-l_3^2)+2\varepsilon^2(2+3l_3^2)]w^4 + 2\{l_3^4 + 2\varepsilon[l_2 - l_3(2+l_3^2)+\\&+\ \varepsilon l_2^2/2]\}w^5 + \ \dots\ + \ w^8 = 0\\&l_2 = 1 - al_1, \quad l_3 = 1 + \varepsilon(1-al_1)\end{aligned} \tag{1.7}$$

where the dots denotes omitted terms involving the sixth and seventh powers in w.

It can be seen from (1.6) and (1.7) that the coefficients of the powers of w are polynomials in ε and, moreover, that these coefficients vanish when $\varepsilon = 0$ for powers of w which are less than four. Consequently, according to Weierstrass' theorem, the representation [5] $f(w, \varepsilon) = \Psi(w, \varepsilon)H(w, \varepsilon)$ is possible, where $f(w, \varepsilon)$ is the polynomial (1.7), $\Psi(w, \varepsilon)$ and $H(w, \varepsilon)$ are algebraic functions of orders three and five respectively and $H(0, 0) \neq 0$. This means that $f(w, \varepsilon)$ has the same roots, which vanish when $\varepsilon = 0$ as $\Psi(\omega, \varepsilon)$ Hence, a solution of Eq. (1.7), which vanishes when $\varepsilon = 0$, can be sought in the form of a series

$$w = \sum_{n=1}^{\infty} a_n \varepsilon^{mn/4}, \quad m = 1,2,3,4 \tag{1.8}$$

In order to select the necessary value of m, let us investigate some properties of the solution of the problem.

Firstly, in the case of the curvature of the extremal, a simple connection with the coordinates

$$k = (\rho^2 + 2\rho'^2 - \rho\rho'')(\rho^2 + \rho'^2)^{-3/2} = c_1/(\rho^2 \sin^2\varphi) - \lambda \tag{1.9}$$

can be established using Eq. (1.3) and the first integral (1.4).

Secondly, it can be shown that, on substituting series (1.8), with $m \neq 4$, into (1.7), the lowest power of ε in series (1.8) for which the coefficient $a_k \neq 0$, is greater than unity. However, a series of the form of (1.8), which represents a real solution of Eq. (1.7) which vanishes when $e = 0$, cannot begin with a power of ε which is greater than unity.

In fact, let $w = a_n\varepsilon^{mn/4} + a_{n+1}\varepsilon^{m(n+1)/4} + \dots$.

According to (1.9), at the point where the extremal touches the wall, the curvature of the extremal is determined by the expression

$$k = k_c = \varepsilon a(1-w)/(a^2 w) + 1/a - \varepsilon l_1 = (1 + \varepsilon^{1-mn/4}/a_n)/a + o(\varepsilon^{1-mn/4})$$

from which it follows that, when $mn/4 - 1 > 0$ and $\varepsilon \to 0$, we will have that $|k_c| \to \infty$.

The Laplace condition [1], $\Delta p = \sigma(k_1 + k_2)$ for the pressure drop, where σ is the surface tension and k_1 and k_2 are the curvatures of the principal normal cross-sections, is satisfied on the surface of separation of the fluid and gas. In the case of a small deformation of the drop, the pressure drop must only change slightly and the convexity of the surface must be preserved. However, this is impossible when $|k_1| = |k_c| \to \infty$. Consequently, $mn/4 - 1 \leq 0$.

The form of series (1.8) is refined in accordance with this

$$w = a_1\varepsilon + a_2\varepsilon^2 + \ \dots \tag{1.10}$$

A comparison of the coefficients of the same powers of ε in Eq. (1.7) after the latter series has been substituted into it leads to an equation for determining the coefficient a_1

$$1 - 4a_1 + 2a_1^2 + 4a_1^3 + a_1^4 = 0 \tag{1.11}$$

This has two real roots: $a_1^+ \approx 2.24$; $a_1^- \approx -0.27$. (Here, they are calculated to two places of decimals.)

The root a_1^+ is the first coefficient of series (1.10).

Actually, by virtue of the non-negative nature of w which is being determined, the signs of a_1 and ε must be the same. Let us prove that $\varepsilon \geqslant 0$.

Direct calculation of the curvature using (1.9) with $\varphi = \pi/2$ and taking account of the fact that $\rho(\pi/2) = h > a$, $\rho'(\pi/2) = 0$, $\rho''(\pi/2) = 2h(1+\lambda h)$ leads to the expression $k = k_0 = -1/h - 2\lambda$. On eliminating λ from this expression and from the right-hand side of (1.9) with $\varphi = \pi/2$, we obtain $c_1 = (k_0 h - 1)/2$. Allowing for the fact that the pressure drop Δp increases when the drop is compressed, we obtain $k_0 + 1/h \geqslant 2/R$ from the Laplace condition when $\varphi = \pi/2$ and $k_1 = k_0$ and $k_2 = 1/h$. Consequently, $c_1 \geqslant h(h/R - 1) \geqslant 0$, $\varepsilon \geqslant 0$ and $a_1 > 0$.

The coefficients of the series, which follow after a_1, are uniquely defined. For example

$$a_2 = (a_1/2)[1 - 2al_1 - 2(1 - 3al_1)a_1 - (5 + 2al_1)a_1^2]/(3a_1^2 + a_1 - 1)$$

We will now consider the problem of integrating Eq. (1.4) by rewriting it, after the introduction of a new variable $z = \sin\varphi$, in the form

$$\frac{d\rho}{dz} = \frac{\rho\{z[\rho^2 z^2 - \xi^2(\rho,z)]^{1/2} + \sqrt{1-z^2}\,\xi(\rho,z)\}}{\sqrt{1-z^2}\,\{\sqrt{1-z^2}\,[\rho^2 z^2 - \xi(\rho,z)]^{1/2} - z\xi(\rho,z)\}} \tag{1.12}$$

$$\xi(\rho,z) = \lambda\rho^2 z^2 + c_1$$

The denominator of the right-hand side of Eq. (1.12) vanishes at the point

$$z = z_a = \{[(1/2 - \lambda c_1)\rho - [(1/4 - \lambda c_1)\rho^2 - c_1^2]^{1/2}]/[(1 + \lambda^2\rho^2)\rho]\}^{1/2} \tag{1.13}$$

that is, at the fixed critical algebraic singular point of Eq. (1.12) if the latter is considered in the space of the complex variables z and ρ [6].

In the neighbourhood of the critical point, the solution of Eq. (1.12) can be represented in the form of certain power series. However, since it is a real solution of this equation which is of interest, it is necessary to show that the point z_a belongs to the domain of definition of the right-hand side of Eq. (1.2) in the case of real z and ρ, that is, that $z_a \geqslant z_{12}$, where $z_{12} = [(1 - 2\lambda c_1 - \sqrt{(1 - 4\lambda c_1)})/(2\lambda^2\rho^2)]^{1/2}$ is a zero of the expression $\rho^2 z^2 - \xi^2(\rho,\ z)$ such that $z_{12} = 0$ when $c_1 = 0$. For sufficiently small ε, we have $z_{12}^2 \leqslant \varepsilon^2(1 - 4\varepsilon) + o(\varepsilon^3)$, $z_a^2 \geqslant \varepsilon^2(1 - 2\varepsilon) + o(\varepsilon^3)$. Moreover, by taking account of the fact that $a/\sqrt{2} < \rho < \sqrt{2}a$ for all φ, the inequality $z_a \leqslant \sqrt{3}\varepsilon$ can be established.

Hence, $z_{12} \leqslant z_a \leqslant z_c$.

Since the right-hand side of Eq. (1.12), raised to a power of minus one, is holomorphic in the neighbourhood of the point $(z_a,\ \rho)$, a solution can be sought in this neighbourhood in the form of a series in powers of $(z - z_a)^{n/k}$ [6], where the number k is equal to the lowest order of the derivative of z with respect to ρ which does not vanish at the point $(z_a,\ \rho)$. Calculations show that, in the case of small ε we have $(d^2z/d\rho^2)_{z=z_a} \neq 0$. We must therefore take $k = 2$, that is, a real solution of (1.12) in the neighbourhood of the point $(z_a,\ \rho)$ can be sought in the form of the series

$$\rho = \rho_a + \sum_{n=1}^{\infty} \alpha_n (z - z_a)^{n/2} \tag{1.14}$$

and z_a is given by expression (1.13) when $\rho = \rho_a$ and ρ_a and α_n are found from the condition

$$\rho_c = \rho_a + \sum_{n=1}^{\infty} \alpha_n (z_c - z_a)^{n/2} \tag{1.15}$$

and from the conditions for the coefficients of like powers of $(z-z_a)^{n/2}$ to be equal after substituting (1.14) into (1.12). For example

$$\alpha_1=\rho_a\{2z_a/[1-z_a^2-2\lambda(\lambda\rho_a^2z_a^2+c_1)]\}^{1/2} \tag{1.16}$$

or

$$\alpha_1=\sqrt{2z_z}\rho_a+o(\sqrt{\varepsilon}) \tag{1.17}$$

On combining (1.17) with the expression $\rho_c-\rho_a=\alpha_1\sqrt{(z_c-z_a)}+o(\sqrt{\varepsilon})$, obtained from (1.15), and taking account of the inequality $z_a\leqslant\sqrt{(3)}\varepsilon$ in choosing the sign in front of the root, we obtain

$$z_a=\tfrac{1}{2}z_c\{1-[1-2(\rho_c-\rho_a)^2/(\rho_az_c)^2]^{1/2}\} \tag{1.18}$$

By eliminating z_a from (1.13) and (1.18) and making simplifications to within terms of the order of $o(\varepsilon^4)$, we obtain the following equation for determining ρ_a

$$\begin{gathered}(1-2z_c+2z_c^2)\chi^4+4z_c(1-z_c)\rho_c\chi^3+4c_1^2(1-z_c)\chi^2+\\+8c_1^2z_c\rho_c\chi-4c_1^2(z_c\rho_c^2-c_1^2)=0\quad(\chi=\rho_c-\rho_a)\end{gathered} \tag{1.19}$$

Noting that, in the case of all small powers of χ up to the third inclusive, the coefficients vanish when $\varepsilon=0$, we seek a solution of Eq. (1.19) which vanishes when $\varepsilon=0$ in the form of the series

$$\chi=\sum_{n=1}^{\infty}\chi_n\varepsilon^{kn/4},\quad k=1,2,3,4$$

It can be shown by direct verification that $k=3$ satisfies the conditions of the problem. Here $\chi_1=a_1^{1/4}\rho_c$.

After calculating $z_a=\varepsilon/2+o(\varepsilon)$ and $\rho_a=a(1-a_1^{1/4}\varepsilon^{3/4}+a_1\varepsilon/2-o(\varepsilon))$, we obtain the asymptotic representation of (1.14)

$$\rho=a(1-a_1^{1/4}\varepsilon^{3/4}+a_1\varepsilon/2)(1+\sqrt{\varepsilon}\sqrt{z-\varepsilon/2})+o(\sqrt{\varepsilon(z-\varepsilon/2)}) \tag{1.20}$$

The series (1.14) defines a curve which begins at the point A (Fig. 1) with the coordinates $z=z_a=\sin\varphi_a$, $\rho=\rho_a$ which, after touching the line $x=a$ at the point C with the coordinates $z=z_c=\sin\varphi_c$, $\rho=\rho_c$, forms the right-hand segment of the extremal. However, the radius of convergence of this series does not include the point $z=1$ at which the left boundary condition $\rho'(\pi/2)=0$ is given. In order to determine the left segment of the extremal, we return to Eq. (1.4) and rewrite it after making the substitution $\cos\varphi=\zeta$

$$\frac{d\rho}{d\zeta}=\frac{\rho\{\zeta\xi(\rho,\zeta)+\sqrt{1-\zeta^2}[\rho^2(1-\zeta^2)-\xi^2(\rho,\zeta)]^{1/2}\}}{(1-\zeta^2)\xi(\rho,\zeta)-\zeta\sqrt{1-\zeta^2}[\rho^2(1-\zeta^2)-\xi^2(\rho,\zeta)]^{1/2}} \tag{1.21}$$

Since the right-hand side of (1.21) is holomorphic for $\zeta^2<1-z_a^2$, a solution of Eq. (1.21) can be sought in the form of the series

$$\rho=h+\sum_{n=1}^{\infty}\beta_{2n}\zeta^{2n} \tag{1.22}$$

which converges for all $\zeta\in[0,\ \zeta_c]$, $\zeta_c=\arccos\varphi_c$. (The evenness of the powers of ζ follows from the symmetry of the problem and the boundary condition $\rho'(\pi/2)=0$ is automatically satisfied here.)

All of the β_{2n} are calculated after substituting the series (1.22) into Eq. (1.21) and comparing the coefficients of like powers of ζ. For example, $\beta_2 = -a\varepsilon/4 + o(\varepsilon)$.

In order to determine the parameters ε and l_1, we make use of the condition for joining the left and right segments of the extremal at a certain value $\varphi = \varphi_b \in [\varphi_c, \pi/2]$, that is, the condition for the right-hand sides of expressions (1.14) and (1.22) to be equal when $\varphi = \varphi_b$

$$\sum_{k=1}^{\infty} \alpha_k (z_b - z_a)^{k/2} = h + \sum_{n=1}^{\infty} \beta_{2n} \zeta_b^{2n} \quad (z_b = \sin\varphi_b,\ \zeta_b = \cos\varphi_b) \tag{1.23}$$

and the condition for the volume of the fluid $(4/3)\pi R^3 = V_0 + V_1 + V_2$ to be constant, where $V_0 = 2\pi a^3 z_c^2/[3(1 - z_c^2)]$ is the volume of a pair of cones with vertices at the point $\rho = 0$ and with bases which coincide with the areas of contact between the fluid and the walls

$$V_1 = 2\pi \int_{\varphi_c}^{\varphi_b} F_0 d\varphi = 2\pi \int_{z_c}^{z_b} F_1 dz, \quad V_2 = 2\pi \int_{\varphi_b}^{\pi/2} F_0 d\varphi = 2\pi \int_0^{\zeta_b} F_2 d\zeta$$

$$F_0 = \rho^3 \sin^3\varphi - \rho' \rho^2 \cos\varphi \sin^2\varphi, \quad F_1 = \rho^2[\rho - (1 - z^2) d\rho/dz] z^2 / \sqrt{1 - z^2}$$

$$F_2 = \rho^2(\rho + \zeta d\rho/d\zeta)(1 - \zeta^2)$$

where ρ, in the case of F_1, is represented by the series (1.14) and, in the case of F_2, by the series (1.22).

A value of φ_b, which is close or equal to a magnitude of $\pi/4$, has to be chosen for the above-mentioned series to converge rapidly. Here also, confining ourselves to a rough estimate of the parameters ε and l_1, we take $\varphi_b = \varphi_c$. The left-hand side in (1.23) can then be replaced by the quantity $a/\sqrt{(1 - z_c^2)}$.

The simplified conditions take the form

$$a(1 + a_1\varepsilon/2) = a[1 + \varepsilon(1 + al_1) - \varepsilon/4] + o(\varepsilon) \tag{1.24}$$

$$(4/3)\pi R^3 = (2/3)\pi a^3 a_1 \varepsilon + (4/3)\pi a^3[1 + \varepsilon(11/4 + 3al_1)] + o(\varepsilon)$$

From (1.24), we find

$$l_1 = (a_1 - 3/2)/(2a), \quad \varepsilon = 2(R^3/a^3 - 1)(4a_1 + 1)$$

2. Let us consider the case when a fluid drop is in contact with the walls with a known angle of contact or, more precisely, in the case of the boundary conditions (Fig. 2)

$$y'(0) = 0, \quad y'(a) = -g, \quad g = \mathrm{ctg}\,\theta, \quad 0 < \theta \leqslant \pi/2 \tag{2.1}$$

The value of g is assumed to be small and θ is the angle of contact.

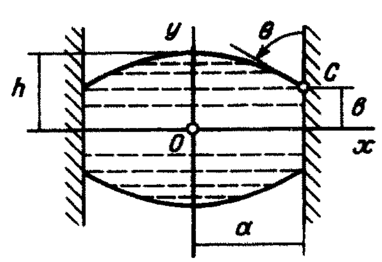

Fig. 2.

It is simpler to solve this problem in Cartesian coordinates. In these coordinates, the first integral has the form

$$y' = -[y^2/(c_1+\lambda y^2)^2 - 1]^{1/2} \tag{2.2}$$

On integrating it, we obtain

$$x = \int (y^2 + c_1/\lambda)[-y^4 + (1/\lambda^2 - 2c_1/\lambda)y^2 - c_1^2/\lambda^2]^{-1/2}dy + c_2 \tag{2.3}$$

$$(0 \le x \le a,\ \ 0 < y,\ \ c_2 = \text{const})$$

Here, a real solution is only possible if the following inequality is satisfied

$$c_1 < 1/(4\lambda) \tag{2.4}$$

In this case, the solution is represented by the elliptic integral

$$x = \int (y^2 + c_1/\lambda)/\sqrt{(y^2 - A^2)(h^2 - y^2)}dy + c_2 \tag{2.5}$$

In the class of convex solutions $y(x) \le h = y(0)$. Hence, instead of (2.5), one can write the expression

$$x = -[(1+\lambda h)/\lambda]F(\Psi, k) + hE(\Psi, k)$$
$$\Psi = \arcsin\sqrt{(h^2 - y^2)(h^2 - A^2)},\quad k^2 = (1 - 2\lambda h)/(\lambda^2 h^2)\quad (\lambda \le (-1/(2h))) \tag{2.6}$$

where $F(\Psi, k)$ and $E(\Psi, k)$ are incomplete elliptic integrals of the first and second kind.

In order to determine the parameters h and λ we make use of conditions (2.1) which, when account is taken of (2.2), reduce to the expression

$$b = [1 - \sqrt{1 - 4c_1\lambda(1+g^2)}]/(2\lambda\sqrt{1+g^2})$$

and the constant-volume condition with is now more conveniently represented in the form

$$V = 2\pi(ab^2 + \int_b^h xy dy) \tag{2.7}$$

In order to close the system of equations in the unknown h, λ and b, it is necessary to make use of relationship (2.6) when $y(a) = b$, that is

$$a = \int_b^h \frac{t^2 + c_1/\lambda}{\sqrt{(t^2 - A^2)(h^2 - t^2)}}dt \tag{2.8}$$

In the general case, it is difficult to obtain explicit expressions for the parameters of the problem from the last three equations. However, in the case when the magnitude of g is small, this can be done.

Let us consider this case. When $g = 0$, the equation of the extremal (1.3) has the solutions $y = b = \text{const}$, if

$$\lambda = \frac{-1}{2b},\quad c_1 = \frac{-1}{2}b\quad \left(b = \sqrt{\frac{V}{2\pi a}}\right)$$

We will seek a solution of Eq. (1.3) in the case of a sufficiently small value of $g = \varepsilon$ in the form of the series

$$y = b + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \dots \tag{2.9}$$

while also representing the values of the parameters λ and c_1 in the form of the series

$$\lambda = \frac{-1}{2b} + \varepsilon l_1 + \varepsilon^2 l_2 + \dots; \quad c_1 = \frac{-1}{2} b + \varepsilon c_{11} + \varepsilon^2 c_{12} + \dots \tag{2.10}$$

As a result of substituting series (2.9) and (2.10) into Eq. (2.2) and regrouping the terms in increasing powers of ε under the radical in (2.2), a new power series of the form

$$A_1\varepsilon + A_2\varepsilon^2 + A_3\varepsilon^3 + \dots, \quad A_1 = (c_{11} + b^2 l_1)/b \tag{2.11}$$

$$A_2 = \frac{1}{b}(3c_{11} + 2C_{12}) + b(3l_1 + 2l_2) + 2\left(l_1 - \frac{1}{b^2}c_{11}\right)\eta_1 - \frac{1}{b^2}\eta_1^2$$

is obtained, that is, Eq. (2.2) takes the form

$$\varepsilon\eta_1^1 + \varepsilon^2\eta_2^1 + \dots = -\sqrt{\varepsilon}\sqrt{A_1 + \varepsilon A_2 + \varepsilon^2 A_3 + \dots} \quad (\eta_k^1 = d\eta/dx) \tag{2.12}$$

In order that this should have a solution, it is necessary that the equality $A_1 = 0$ should be satisfied. Obviously, $A_2 \neq 0$. We will seek a solution of (2.12) for which $A_2 > 0$ when $\eta_1 = 0$. Equation (2.12) can then be written in the form of the series

$$\eta_1' + \varepsilon\eta_2' + \dots = -\sqrt{A_1 + \varepsilon A_3 + \varepsilon^2 A_4 + \dots} = -\sqrt{A_2}(1 + \varepsilon B_2 + \varepsilon^2 B_3 + \dots) \tag{2.13}$$

which converges in a certain neighbourhood of $\varepsilon = 0$.

From a comparison of the coefficients of like powers of ε in (2.13), we obtain a sequence of differential equations

$$\eta_1' = -\sqrt{A_2}; \quad \eta_k' = -\sqrt{A_2} B_k; \quad k = 2,3,\dots \tag{2.14}$$

with the boundary conditions

$$\eta_1'(0) = 0; \quad \eta_1'(a) = -1; \quad \eta_k'(0) = 0; \quad \eta_k'(a) = 0, \quad k > 1 \tag{2.15}$$

which are ensured by the choice of constants c_{1k} and c_{2k}, where c_{2k} are constants which appear when integrating (2.14). Actually, these constants, together with the constants l_k, are determined from conditions (2.15) and the constant-volume condition

$$\int_0^a [2(b + \varepsilon\eta_1 + \dots + \varepsilon^{k-1}\eta_{k-1})\eta_k + \varepsilon^k \eta_k^2] dx = 0, \quad k = 1,2,\dots \tag{2.16}$$

Without touching here on the question of the solvability of the boundary-value problem for all k, let us consider the solution when $k = 1$. When $k = 1$ and allowing for the fact that $A_1 = 0$ or $c_{11} + b^2 l_1 = 0$, we have

$$\eta_1' = -\left[\frac{2}{b}c_{12} + 2bl_2 + 4l_1\eta_1 - \frac{1}{b^2}\eta_1^2\right]^{1/2} \tag{2.17}$$

and, after integration

$$\eta_1 = 2b^2[l_1 - \sqrt{l_1^2 + l_{12}} \sin(x/b + c_{21})], \quad l_{12} = (c_{12}/b^2 + l_2)/(2b)$$

$$c_{21} = \text{const}$$

The condition $\eta'(0) = 0$ can be satisfied by taking $c_{21} = -\pi/2$. Then

$$\eta_1 = 2b^2[l_1 + \sqrt{l_1^2 + l_{12}}\cos(x/b)]$$

From the other boundary condition $\eta_1'(a) = -1$, we obtain $\sqrt{(l_1^2 + l_{12})} = 1/[2b\sin(a/b)]$ and, from condition (2.16) or $\int_0^a \eta_1 dx + o(\varepsilon) = 0$, we obtain $l_1 = -(b/a)\sqrt{(l_1^2 + l_{12})}\sin(a/b)$.

On solving the resulting relationships for c_{11}, l_1 and l_{12}, we find $c_{11} = b^2/(2a)$, $l_1 = -1(2a)$, $l_{12} = 1/[4b^2\sin^2(a/b)] - 1/(4a^2)$ subject to the condition that $0 < a < \pi b$.

The parameters which have been found satisfy the previously assumed inequality $A_2 > 0$ when $\eta_1 = 0$, since $\sin(a/b) < a/b$ and, consequently $l_{12} > 0$. Finally, the asymptotic representation of the extremal will be

$$y = b + \varepsilon b^2\{-1/a + \cos(x/b)/[b\sin(a/b)]\} + o(\varepsilon)$$

$$\lambda = -1/(2b) - \varepsilon/(2a) + o(\varepsilon), \quad c_1 = -b/2 + \varepsilon b^2/(2a) + o(\varepsilon) \quad (b = \sqrt{V/(2\pi a)})$$

In applications, the magnitude of the force due to the pressure of the drop on the wall may be of interest. This can be calculated by determine the area of contact and the pressure drop (on the equator of the drop, for example) using the Legendre condition.

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